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MATHEMATICS
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On the Predicate Calculi With Iteration Quantifiers

by

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In this paper we investigate the deductive systems and theories whose formalized languages are certain sublanguages of algorithmic languages as introduced in [2]. The completeness theorem as well as the theorem on the existence of models for consistent theories will be proved here with the use of algebraic methods. We assume the reader to be familiar with the concepts introduced in [2].

1. Languages and their realizations

The alphabet A of a language $\mathcal{L}$ is obtained by adding auxiliary symbols to the alphabet of a language $\mathcal{L}_0$ of open formulae. These are: signs of iteration quantifiers $\bigcup$ and $\bigcap$, square brackets [ ], and an auxiliary sign /.

The set T of terms is defined as usual. The set S of substitutions will be composed of all expressions of the form $[x_1/\tau_1 \ldots x_n/\tau_n]$, x_i being an individual variable and τ_i a term.

The set of formulae will be the least set F satisfying:

f1) if ρ is an n-argument predicate and $\tau_1, \ldots, \tau_n$ are terms, then $\rho(\tau_1 \ldots \tau_n)$ is in F,

f2) if α is in F, then $\neg\alpha$ is in F,

f3) if α and β are in F, then $(\alpha \vee \beta)$, $(\alpha \wedge \beta)$, $(\alpha \Rightarrow \beta)$ are in F,

f4) if α is in F and s is a substitution, then $s\alpha$, $\bigcup s\alpha$, $\bigcap s\alpha$ are in F.

Elements of the set F will be denoted by $\alpha, \beta, \gamma, \delta$ (with indices if necessary). It is easy to see that the set F^0 of open formulae is a subset of F. Let $s^i\alpha$ be an abbreviation for a formula defined by induction as follows

$$s^0\alpha = \alpha,$$

$$s^{i+1}\alpha = ss^i\alpha.$$

The formula α of the form mentioned in f1) will be called atomic. By the language $\mathcal{L}$ with iteration quantifiers we understand an ordered system $\langle A, T, S, F\rangle$. The class of all languages with iteration quantifiers will be denoted by $\boldsymbol{L}$.

1.1. The set F of all formulae of any language $\mathcal{L} \in L$ is an abstract algebra

$$\langle F, \neg, \vee, \wedge, \Rightarrow \bigcup, \bigcap \{s\}_{s \in S}\rangle$$

with one 1-argument operation $\neg$, three 2-argument operations $\vee, \wedge, \Rightarrow$ if formulae $\neg a$, $(a \vee \beta)$, $(a \wedge \beta)$, $(a \Rightarrow \beta)$ are conceived as the results of corresponding operations on the formulae a and β. The set S of all substitutions forms the set of operators on the set F of all formulae. An operator s applied to a formula a gives the formula sa. Formulae $\bigcup sa$ and $\bigcap sa$ are treated as the results of infinite operations $\bigcup$ and $\bigcap$ on the set of all formulae of the form $s^i a$ $(i=0, 1, \ldots)$.

Common domain $\mathcal{D}$ of the infinite operations $\bigcup$ and $\bigcap$ is the family of sets of formulae such that the set $G \subset F$ is in $\mathcal{D}$ iff there exist a formula a and a substitution s such that the set G is composed of all formulae of the form $s^i a$ $(i=0, 1, \ldots)$.

1.2. The algebra $\langle F, \neg, \vee, \wedge, \Rightarrow, \bigcup, \bigcap, \{s\}_{s \in S}\rangle$ is a generalized free algebra for the class $\mathcal{K}$ of all similar algebras

$$\langle A, o_1, o_2, o_3, o_4, O_1, O_2, \{\sigma_s\}_{s \in S}\rangle$$

with one 1-argument operation o_1, three 2-argument operations o_2, o_3, o_4 two infinite operations O_1 and O_2 and the set of operators $\{\sigma_s\}_{s \in S}$. The set composed of all atomic formulas is the set of free generators of the algebra F.

We omit the easy proofs.

The language $\mathcal{L} = \langle A, T, S, F\rangle$ is a sublanguage of an algorithmic language as defined in [2]. Any realization of an algorithmic language will be considered as a realization of a corresponding language.

Let τ be a term, and $VF(\tau)$ the set of individual variables occurring in τ.

Let a be a formula, by $VF(a)$ we shall denote the set of individual variables occurring free in a.

v1) If a is atomic, then $VF(a)$ contains all variables occurring in a,

v2) if a is of the form $\neg \beta$ then $VF(a) = VF(\beta)$,

v3) if a is one of the forms $(\beta \vee \gamma)$, $(\beta \wedge \gamma)$, $(\beta \Rightarrow \gamma)$ then

$$VF(a) = VF(\beta) \cup VF(\gamma).$$

Let s be a substitution $[x_1/\tau_1 \ldots x_n/\tau_n]$ then the set of variables occurring as the first element in a pair x_i/τ_i will be denoted by $OR(s)$. The set of all variables occurring in any term τ_i will be denoted by $IR(s) = \bigcup_{i=1} VF(\tau_i)$. Let $\overline{sx_i}$ denote the term. τ_i.

v4) If a is of the form $s\beta$ then

$$VF(a) = \big(VF(\beta) - OR(s)\big) \cup \bigcup VF(\overline{sx})\ x \in VF(\beta) \cap OR(s),$$

v5) if a is of the form $\bigcup s\beta$ or $\bigcap s\beta$ then

$$VF(a) = \bigcup_{i=0}^{\infty} VF(s^i \beta).$$

Observe that there exists an integer k such that $\bigcup_{i=0}^{\infty} VF(s^i \beta) = \bigcup_{i=0}^{k} VF(s^i \beta)$ $(k \leqslant n)$.

Let θ be a term or an open formula. Let s be a substitution $[x_1/\tau_1, ..., x_n/\tau_n]$. By $\overline{sx}$ we denote τ_i if $x=x_i$ or x if x differs of all x_i $(i=1, ..., n)$. Let us consider the set of individual variables $V_{s\theta}=\{x \in V: x \in VF(\theta) \cap OR(s)\}$. By $\overline{s\theta}$ we denote the result of simultaneous replacement of all occurrences of individual variables x from $V_{s\theta}$ by $\overline{sx}$. We recall [see .1, V. 3] that the result $\overline{s\theta}$ will be a term (an open formula) if θ is a term (an open formula).

From the definition of realization of terms, formulae and substitutions [see 2] follows immediately

1.3. The value $a_R(v)$ of the formula a in the realization R at the valuation v depends only on $\varphi_{1R}, ..., \varphi_{nR}, \rho_{1R}, ..., \rho_{mR}$ and $v_{x_1}, ..., v_{x_k}$,
where: $\varphi_1, ..., \varphi_n$ are all functors occurring in the formula a
$\rho_1, ..., \rho_m$ are all predicates occurring in the formula a
$x_1, ..., x_k$ are all free variables occurring in the formula a.

1.4. For every atomic formula γ, every formulae a, β, every substitutions s and r and every valuation v the following equalities hold

$$(s\gamma)_R(v)=\overline{s\gamma}_R(v),$$
$$(s(a \vee \beta))_R(v)=(sa)_R(v) \cup (s\beta)_R(v),$$
$$(s(a \wedge \beta))_R(v)=(sa)_R(v) \cap (s\beta)_R(v),$$
$$(s(a \Rightarrow \beta))_R(v)=(sa)_R(v) \rightarrow (s\beta)_R(v),$$
$$(s \neg a)_R(v)=-(sa)_R(v),$$
$$(r \bigcup sa)_R(v)=\bigcup_{i=0}^{\infty} (rs^i a)_R(v),$$
$$(r \bigcap sa)_R(v)=\bigcap_{i=0}^{\infty} (rs^i a)_R(v).$$

1.5. If in any Boolean algebra $\mathfrak{B}=\langle B, \cup, \cap, \rightarrow, \bigcap, \bigcup\rangle$ the results of corresponding infinite operations exist, then

$$\bigcup_{t \in T \cup \{t_0\}} a_t = a_{t_0} \cup \bigcup_{t \in T} a_t,$$
$$\bigcap_{t \in T \cup \{t_0\}} b_t = b_{t_0} \cap \bigcap_{t \in T} b_t.$$

As a simple corrolary we obtain

1.6. For every formula a every substitutions r and s and every valuation the following equalities hold

$$(r \bigcap sa)_R(v)=(ra)_R(v) \cap (r \bigcap s\, sa)_R(v),$$
$$(r \bigcup sa)_R(v)=(ra)_R(v) \cup (r \bigcup s\, sa)_R(v).$$

2. Axioms of the calculus

By an axiom we understand any formula of the form (t1)—(t12) or (p1)—(p7). The axioms (t1)—(t12) of clasical propositional calculus are taken from Rasiowa and Sikorski [1]. The second group of axioms is as follows. Let us assume the following abbreviation: by $(a \Leftrightarrow \beta)$ we shall understand the following formula $((a \Rightarrow \beta) \wedge (\beta \Rightarrow a))$. With this abbreviation, any formula of the one of the following schemas is an axiom

(p1) $(s\gamma \Leftrightarrow \overline{s\gamma})$,

(p2) $(s(a \vee \beta) \Leftrightarrow (sa \vee s\beta))$,

(p3) $(s(a \wedge \beta) \Leftrightarrow (sa \wedge s\beta))$,

(p4) $(s(a \Rightarrow \beta) \Leftrightarrow (sa \Rightarrow s\beta))$,

(p5) $(s \neg a \Leftrightarrow \neg sa)$,

(p6) $(r \bigcup sa \Leftrightarrow (ra \cup r \bigcup s\, sa))$,

(p7) $(r \bigcap sa \Leftrightarrow (ra \cap r \bigcap s\, sa))$

for every formulae a every atomic formula γ, every substitutions s and r, every predicate ρ, and every term $\tau_1, \ldots, \tau_n$.

We assume the following rules of inference

r1) rule of *modus ponens*

$$\frac{a, (a \Rightarrow \beta)}{\beta},$$

r2) rule of substitution

$$\frac{a}{sa},$$

r3) rule of generalization

$$\frac{a}{\bigcap sa},$$

r4) rule of introduction of a universal quantifier of iteration

$$\frac{\{(a \Rightarrow s^i \beta)\}_{i \in N}}{(a \Rightarrow \bigcap s\beta)},$$

r5) rule of introduction of an existential quantifier of iteration

$$\frac{\{(s^i a \Rightarrow \beta)\}_{i \in N}}{\bigcup sa \Rightarrow \beta}.$$

Rule r1) has two premises, rules r2) and r3) have one premise. Rules r4) and r5) have infinitely many premises. However in the case when the set $OR(s)$ is disjoint with the set of free variables of the formula a (of the formula β) the rule r4 (r5) can be replaced by a simpler one

r4') $\dfrac{(a \Rightarrow \beta)}{(a \Rightarrow \bigcap s\beta)}$ if $OR(s) \cap VF(a) = \varnothing$,

r5') $\dfrac{(a \Rightarrow \beta)}{(\bigcup sa \Rightarrow \beta)}$ if $OR(s) \cap VF(\beta) = \varnothing$.

Let G be a set of formulae. By $C(G)$ we denote the least set of formulae containing G and axioms of calculus and closed under the rules of inference r1)—r5). The mapping C assigning to every set G a set $C(G)$ will be called the consequence operation. It is easy to check the following properties of C

c1) $G \subset C(G)$,

c2) $G \subset H$ implies $C(G) \subset C(H)$,

c3) $CC(G) = C(G)$.

In the sequel we shall study the deductive system

$$\mathcal{S} = \langle \mathcal{L}, \mathcal{C} \rangle$$

and theories $\mathcal{T} = \langle \mathcal{L}, \mathcal{C}, \mathcal{A} \rangle$ where $\mathcal{A}$ is a set of formulae called specific axioms of theory $\mathcal{T}$. The set $\mathcal{C}(\mathcal{A})$ will be called the set of theorems of a theory $\mathcal{T} = \langle \mathcal{L}, \mathcal{C}\ \mathcal{A} \rangle$ The deductive system $\mathcal{S}$ can be considered as a theory with empty set of specific axioms.

Notions of consistency of theory $\mathcal{T}$, and of a formula irrefutable in the theory $\mathcal{T}$ are defined as usual.

Let us introduce the following congruence relation in the set F of formulae

$$a \approx \beta \text{ iff formulae } (a \Rightarrow \beta) \quad \text{and} \quad (\beta \Rightarrow a) \quad \text{are theorems of theory}$$

[see 1, VI. 10].

2.1. If formulae a and β are congruent $a \approx \beta$, then

$$sa \approx s\beta,$$

$$\bigcap sa \approx \bigcap s\beta,$$

$$\bigcup sa \approx \bigcup s\beta.$$

The quotient algebra $F/\approx$ will be denoted $\mathfrak{A}(\mathcal{T})$ and called Lindenbaum algebra of the theory $\mathcal{T}$. Elements $\mathfrak{A}(\mathcal{T})$ are classes of equivalence $\approx$ denoted by $\|a\|$.

2.2. The Lindenbaum algebra $\langle \mathfrak{A}(\mathcal{T}), \wedge, \wedge, \Rightarrow, \neg, \bigcup, \bigcap, \{s\}_{s \in S} \rangle$ of the theory $\mathcal{T}$ is a Boolean algebra with endomorphisms $\{s\}_{s \in S}$ and with infinite operations of l.u.b. and g.l.b. For any formulae a and β.

$$\|a\| \vee \|\beta\| = \|(a \vee \beta)\|,$$

$$\|a\| \wedge \|\beta\| = \|(a \wedge \beta)\|,$$

$$\|a\| \Rightarrow \|\beta\| = \|(a \Rightarrow \beta)\|,$$

$$\neg \|a\| = \| \neg a\|.$$

Moreover, $\|a\| \leqslant \|\beta\|$, iff the formula $(a \Leftarrow \beta)$ is a theorem of the theory $\mathcal{T}$. The result of the operator s on the element $\|a\|$ is defined by the equality

$$s\,\|a\| = \|sa\|$$

and the following equalities hold

$$s\,\|(\alpha \vee \beta)\| = s\,\|\alpha\| \vee s\,\|\beta\|,$$
$$s\,\|(\alpha \wedge \beta)\| = s\,\|\alpha\| \wedge s\,\|\beta\|,$$
$$s\,\|(\alpha \Rightarrow \beta)\| = s\,\|\alpha\| \Rightarrow s\,\|\beta\|,$$
$$s\,\|\neg\,\alpha\| = \neg\, s\,\|\alpha\|.$$

For every formula α

$\|\alpha\| = \vee$ iff α is a theorem of the theory $\mathcal{T}$,

$\|\alpha\| \neq \wedge$ iff α is irrefutable in the theory $\mathcal{T}$

and algebra $\mathfrak{A}(\mathcal{T})$ is nondegenerated (i.e. has at least two elements) iff the theory $\mathcal{T}$ is consistent.

The equivalence classes of formulae $\bigcup s\alpha$ and $\bigcap s\alpha$ are l.u.b. and g.l.b. of the set of equivalence classes $\{\|s^i\alpha\|\}$ $i=0, 1, \ldots$

$$\|\bigcup s\alpha\| = \bigcup_{i=0}^{\infty} \|s^i\alpha\|,$$
$$\|\bigcap s\alpha\| = \bigcap_{i=0}^{\infty} \|s^i\alpha\|.$$

The proof follows partially from 2.1 and from [1, VII.1.1], the remaining parts being easy to verify.

3. Validity of formulae. Models of a theory

Notions of satisfiability of a formula, of validity and of model are defined as usual.

3.1. Let a realization R of the language $\mathcal{L}$ be a model for the theory $\mathcal{T} = \langle \mathcal{L}, \mathcal{C}, \mathcal{A} \rangle$, then every formula $\alpha \in \mathcal{C}(\mathcal{A})$ (i.e. every theorem of this theory) is valid in R.

Proof by easy verification.

3.2. Every theorem of the deductive system $\mathcal{S}$ is valid in every realization R.

3.3. If the theory $\mathcal{T}$ has a model in a nonempty set J and the two-element Boolean alegbra B_0, then $\mathcal{T}$ is consistent.

3.4. Deductive systeem $\mathcal{S}$ is consistent.

3.5. If a formula of the form $((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha))$ is a theorem of deductive system $\mathcal{S}$, then $\alpha_R(v) = \beta_R(v)$ for every realization R and every valuation v.

4. Completeness theorem

Now we shall state a theorem converse to **3.1.**

4.1. If the theory $\mathcal{T} = \langle \mathcal{L}, \mathcal{C}, \mathcal{A} \rangle$ is consistent, then there exists a set of formulae $\mathcal{G}$ such that

a) $\mathcal{G}$ is consistent, i.e. the theory $\langle \mathcal{L}, \mathcal{C}, \mathcal{G} \rangle$ is consistent,

b) $\mathcal{A} \subset \mathcal{G}$,

c) for every formula a either $a \in \mathcal{G}$ or $\neg a \in \mathcal{G}$

d) if formula $\bigcup sa$ belongs to $\mathcal{G}$, then there exists an integer k such that $s^k\, a \in \mathcal{G}$.

This is esily proved making use of the lemma of Rasiowa—Sikorski [1,II.9.3]

4.2. If a theory $\mathcal{T} = \langle \mathcal{L}, \mathcal{C}, \mathcal{A} \rangle$ is consistent, then there exists a model for $\mathcal{T}$.

4.3. The following conditions are equivalent:

(i) the formula a is a theorem of the consistent theory $\mathcal{T} = \langle \mathcal{L}, \mathcal{C}, \mathcal{A} \rangle$,

(ii) the formula a is valid in every model R of the theory $\mathcal{T}$.

At the first glance it seems possible to weaken the axiomatization presented here by replacing infinitistic rules r4) and r5) by some finitistic rules. That this is not the case will be shown as follows.

Let us consider two languages, first the classical (denoted by $\mathcal{L}_{\omega\omega}$) and the second containing the iteration quantifiers (denoted by $\mathcal{L}$). Both languages have the same functors 0 (zero) and S (successor) and one predicate $=$ (equality). Observe the following facts. 1° It is possible to construct the formulae $a_1, a_2, a_3 \in \mathcal{L}$ such that the theory $Ar = \langle \mathcal{L}, \mathcal{C}, \{a_1, a_2, a_3\} \rangle$ is consistent and all the models of Ar are isomorphic with the set of natural numbers. 2° Let R be a realization of functors and of $=$ in the set of natural numbers. To every formula $a \in \mathcal{L}_{\omega\omega}$ one can assign a formula $\beta \in \mathcal{L}$ such that for every valuation v of individual variables $a_R(v) = \beta_R(v)$. The assumption that it is possible to modify the axiomatization presented above in such a way that proofs are always finite leads to contradiction with the results of Gödel that the arithmetic is undecidable.

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REFERENCES

[1] H. Rasiowa, R. Sikorski, *Mathematics of metamathematics*, Warsaw, 1963.

[2] A. Salwicki, *Formalized algorithmic languages*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronon. Phys., **18** (1970), 227–232