SpecVer project (2026) 1–??
Dombrova Research, September 1, 2026

Collatz conjecture becomes theorem/ Extended abstract

Grażyna Mirkowska
UKSW University, Warsaw
Institute of Informatics

Andrzej Salwicki
Dombrova Research
Partyzantów 19, 05-092 Łomianki, POLAND
A.Salwicki $\boxed{a}$ *mimuw·edu.pl*

Keywords: algorithm of Collatz, halting property, Collatz tree, calculus of programs,algorithmic theory of numbers.

ACM-class: F.3.1; D.2.4
MSC-class: 03D02 (Primary) 68Q02 (Secondary)

Abstract.
We are presenting the paradox, i.e. two theses T1 and T2 that contradict each other. Third thesis T3 solves the problem.

T1. We show that the Collatz conjecture "For every natural number n, the $3n + 1$ computation is finite." is a *semantically valid statement*.

The sufficient and necessary criterion $\varphi(n)$ for termination of $3n + 1$ computation is given 2, c.f. page 2.
We prove that, every instance $\varphi(n/r)$ of the criterion where the variable n is replaced by any natural number $r \neq 0$, is a *theorem* of Peano's arithmetic, Hence, the set $\{\varphi(n/r)\}_{r=0}^{\infty} \subset Th(\mathcal{PA})$ is a recursive subset of the set of theorems.

T2. Paradoxically, the Collatz conjecture itself, *is not a theorem* of number theory (Peano's arithmetic or a similar elementary theory).

It is so because, 1°the formula $\forall_n \varphi(n)$, obtained by putting the general quantifier $\forall_n$ in front of formula $\varphi(n)$, may obtain the value $\mathbb{F}$ - false, in a non-standard model of Peano's arithmetic and 2 °there is no way to bound the classical quantifier to the set of standard, reachable natural numbers.
To avoid the paradox, we will conduct our considerations in the formalized *algorithmic* theory $\mathcal{ATN}$ of natural numbers. The logical consequence operation of the theory is determined by the calculus of programs $\mathcal{AL}$, which is an extension of the predicate calculus.
The halting condition of the computations 2 is expressed by an algorithmic formula, c.f. the consequent of(MainThm).

T3. We are proving that, four infinite sets St_0, St_1, St_2, St_3 of formulas, are the *recursive sets* of *theorems* of the theory $\mathcal{ATN}$. Hence, every formula of the set St_3 has a proof.
The proof of (MainThm) concludes by application of the infinitary inference rule R_3 to the infinite set St_3 of premises.

$$\mathcal{ATN} \vdash \forall_{n>0} \left(\underbrace{\left\{ \begin{array}{l} q \leftarrow 1; \\ \textbf{while}\ n \neq q\ \textbf{do} \\ \quad q \leftarrow q+1 \\ \textbf{od} \end{array} \right\} (n = q)}_{\mathbb{IF}\ n\ is\ a\ natural\ number} \implies \underbrace{\left\{ \begin{array}{l} m \leftarrow \rho(n); \\ \textbf{while}\ m \neq 1\ \textbf{do} \\ \quad m \leftarrow \rho(3m+1) \\ \textbf{od} \end{array} \right\} (m = 1)}_{\mathrm{THEN}\ the\ computation\ for\ n\ is\ finite\ \mathbb{FI}} \right) \tag{MainThm}$$

DEF. The function ρ is defined as $\rho(n) = (2j+1) \overset{df}{\Leftrightarrow} \exists_i \exists_j\, n = 2^i \cdot (2j+1)$.

Extended Abstract

1. Introduction

This is an abbreviated version of the full paper of the same title, that you can find at
https://lem12.uksw.edu.pl/wiki/CollatzConjectureBecomesTheorem.pdf

The most of lemmas and all the proofs were omitted.

2. Three steps reducing problems

In this section we recall the Collatz's conjecture and two statements equivalent to the conjecture.

2.0.1. Collatz tree

Conjecture 1. (of Lothar Collatz)
The Collatz tree, see figure 1, contains all natural numbers.

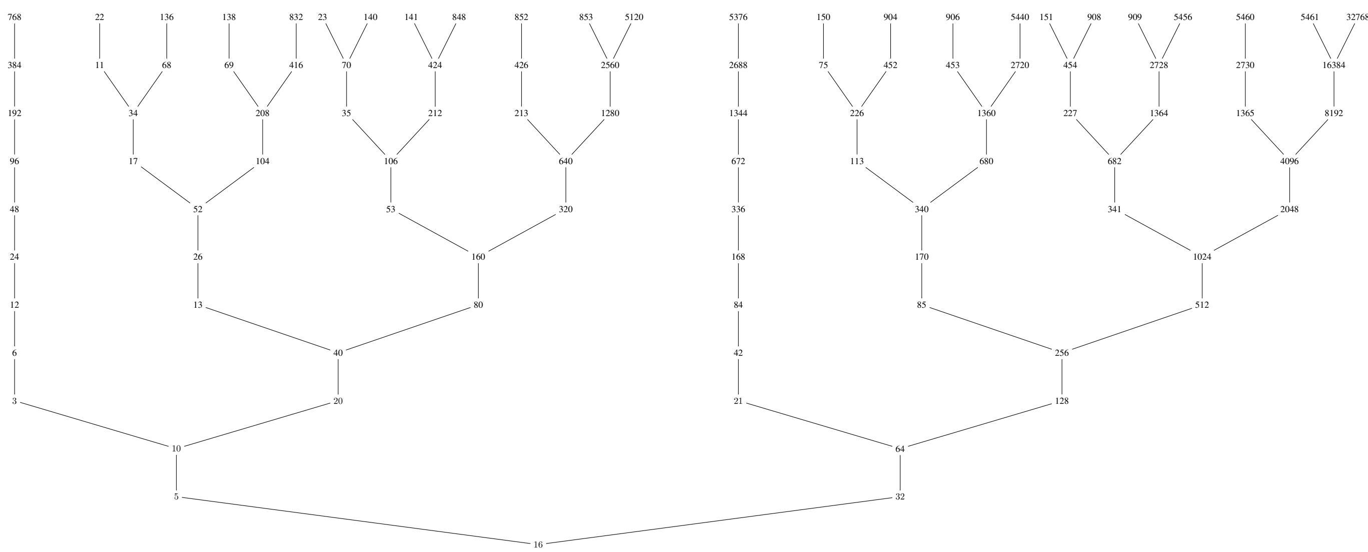

Figure 1. A fragment of Collatz tree

Shown levels 4-15. It does not include levels 0-3, they consist of elements 1 — 2 — 4 — 8 — .
Does every natural number n belong to the Collatz tree?

We need the following definitions of the auxiliary functions $\kappa, \rho\colon N \to N$

$$\text{for every natural number } n,\ \kappa(n) = i \wedge \rho(n) = (2j+1) \equiv \exists i, j\ n = 2^i \cdot (2j+1)$$

Lemma 2.1. (***The sufficient and necessary criterion of termination of*** $3n+1$ ***computations***)
Let r be a natural number (i.e. a numeral). The following conditions are equivalent

(i) the $3n+1$ computation for the number r is finite, (*ii*) the following sentence 1 is valid in $\mathfrak{N}$

$$\exists_x \left(r \cdot 3^x + y = 2^z \wedge \left(\begin{array}{l} \left(y = \sum_{j=0}^{x-1} 3^{x-1-j} \cdot 2^{\sum_{p=0}^{j} k_p} \right) \wedge \left(z = \sum_{p=0}^{x} k_p \right) \wedge \\ \Big(k_0 = \kappa(r) \wedge m_0 = \rho(r) \Big) \wedge \\ \bigwedge_{l=0}^{x-1} \Big(k_{l+1} = \kappa(3m_l + 1) \wedge m_{l+1} = \rho(3m_l + 1) \Big) \end{array} \right) \right). \tag{1}$$

(*iii*) the following sentence 2 is valid in $\mathfrak{N}$

$$\{x := 0\} \bigcup \{x := x+1\} \left((r \cdot 3^x + y = 2^z) \wedge \left(\begin{array}{l} \left(y = \sum_{j=0}^{x-1} 3^{x-1-j} \cdot 2^{\sum_{p=0}^{j} k_p} \right) \wedge \left(z = \sum_{p=0}^{x} k_p \right) \wedge \\ \Big(k_0 = \kappa(r) \wedge m_0 = \rho(r) \Big) \wedge \\ \bigwedge_{l=0}^{x-1} \Big(k_{l+1} = \kappa(3m_l + 1) \wedge m_{l+1} = \rho(3m_l + 1) \Big) \end{array} \right) \right). \tag{2}$$

Perhaps you have noticed the difference between formulas 1 and 2.
Here we are offering a couple of *informal* explanations.
The formula 2 is the correct criterion for the finiteness of 3n+1 computations. The logical value of it says, *informally*, there exists x such that $x = 0 \vee x = 1 \vee x = 2 \vee \ldots$ and the conjunction contained in big parentheses is satisfied by x. On the other hand, the classical quantifier $\forall$ does not provide such comfort. It does not exclude the possibility that the value of x is non-standard. The formula 1 should only appear together with the reachability axiom (R) in the subsection **??**.

2.1. Hotel HC

In the following figure 2 one can find any natural number, like in the picture by Geaorg Cantor.
Some pairs $\langle n, m \rangle$ are connected as described by L. Collatz. Is this graph a tree? If so our problem is solved.

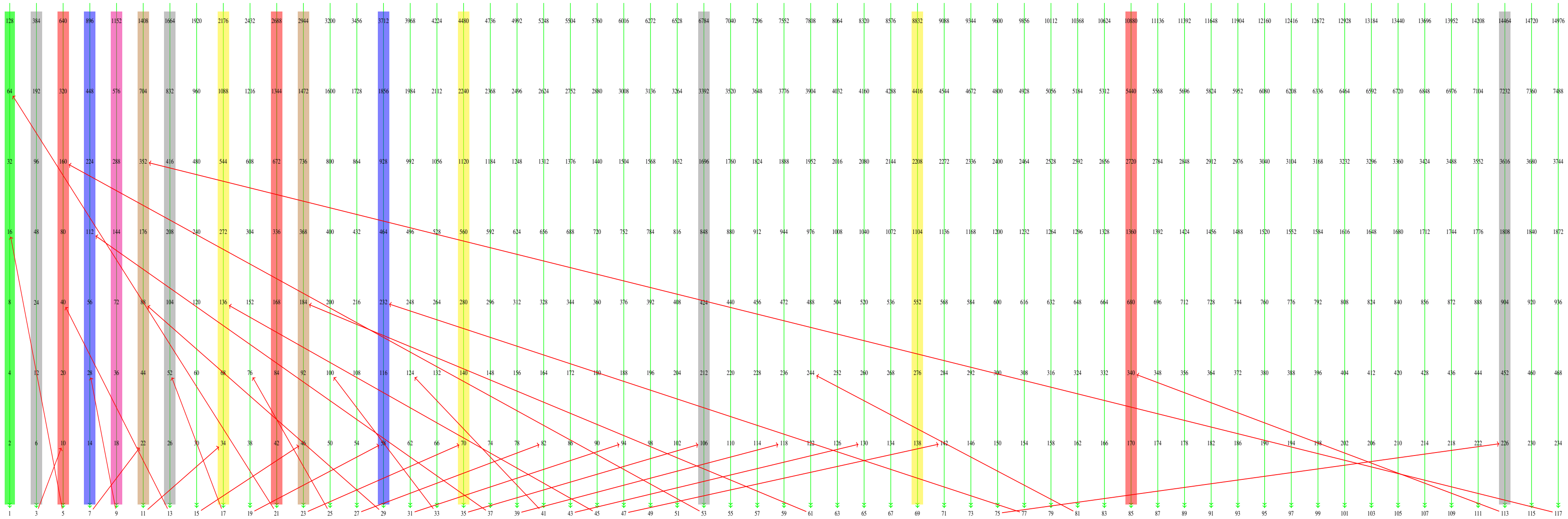

Figure 2. Hotel Collatz

2.2. The graph $\mathcal{G}$ of odd, natural numers

We shall introduce the notions of successors and of predecessor of an odd, natural number. The odd numbers divisible by 3 have no successors. We shall use the following function δ to abbreviate the formulas. For very odd number not divisible by 3, we put $\delta(n) \stackrel{df}{=} (n \mod 3) - 1$.

Definition 2.1. [of successor $\quad m = S_i(n)$]
The i-th successor of an odd number n is defined as follows

$$S_i(n) \stackrel{df}{=} \frac{n \cdot 2^{2i-\delta(n)} - 1}{3}, \qquad \text{where } i = 1, 2, \dots .$$

NOTE, the odd numbers divisible by 3 ($n \mod 3 = 0$) have no successors.
NOTE successor $S_1(1)$ is not defined.

Definition 2.2. (of predecessor $n = P(m)$)
The predecessor of an odd number m is the odd number $n = \dfrac{3 * m + 1}{2^{\kappa(3m+1)}}$.

We define the graph $\mathcal{G}$, To see an example subgraph of it, look at Fig. 3,

Definition 2.3. Graph $\mathcal{G}$ of odd numbers is the system of two sets

$$\mathcal{G} \stackrel{df}{=} \{V, E\}$$

The set V (of *nodes*) is the set of all odd natural numbers.
The set E (of *edges*) is the set of all pairs $\langle n, m \rangle$ such that

c1) n, m are odd natural numbers, $E \subset V \times V$,

c2) the number n is indivisible by 3, $n \bmod 3 \neq 0$,

c3) $m = \cdot \dfrac{n \cdot 2^{2i-\delta(n)} - 1}{3}$ where $i \in \{1, 2, \dots\}$ and $\delta(n) \stackrel{df}{=} (n \bmod 3) - 1$.

W may conceive the graph $\mathcal{G}$ as an algebraic system

$$\mathfrak{G} = \langle V; 1, \{S_i\}_{i=0}^{\infty}, P, = \rangle$$

The following lemma gathers a couple of useful facts. Note, it's similarity to the axioms of Presburger's theory.

Lemma 2.2.

$$S_i(n) \neq 1 \tag{3}$$
$$S_i(n) \neq n \tag{4}$$
$$m = S_i(n) \Rightarrow P(m) = n \tag{5}$$
$$P(m) = n \Rightarrow \exists_{i>0}\, m = S_i(n) \tag{6}$$
$$S_i(n) = S_j(n) \implies i = j \tag{7}$$
$$S_i(n) = S_i(m) \implies n = m \tag{8}$$
$$i \neq j \implies S_j(S_i(n)) \neq S_i(S_j(n)) \tag{9}$$
$$i < j \implies S_i(n) < S_j(n) \tag{10}$$
$$\text{the value of } S_1(1) \text{ is undefined as well as the value of } P(1) \tag{11}$$

Proof:
We shall prove the formula (6). If $P(m) = n$ then $n \cdot 2^{\kappa(3 \cdot m+1)} = 3 \cdot m + 1$. Hence $i = (\kappa(3 \cdot m + 1) + \delta(n)) \div 2$.
The proof of the remaining formulas is left to the reader. □

It suffices to show, that the graph $\mathcal{G}$ is a tree to obtain that the graph $\mathcal{HC}$ is a tree and hence the tree DC contains every natural number.

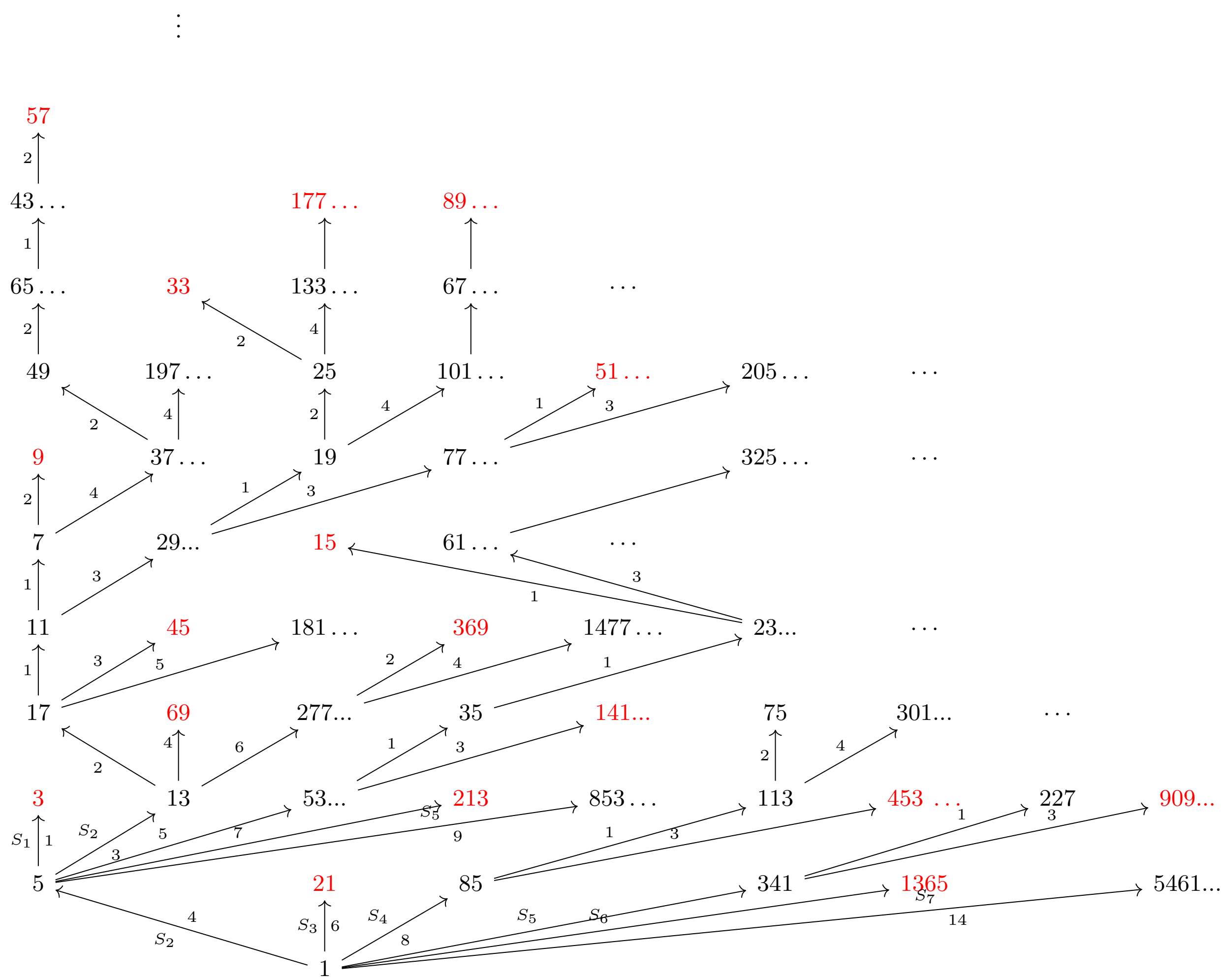

Figure 3. A fragment of the graph $\mathcal{G}$, The numbers k_i on the edges indicate the numbers ofsuccessors S_i, namely $i = \lceil \frac{k_i}{2} \rceil$

3. Arguing that the graph $\mathcal{G}$ is a tree

Our intuition tells us that the graph $\mathcal{G}$ is a tree. However, we need a proof. We are oing to use the trick of Georg Cantor: define an increasing sequence of trees D_c. For each natural number c the tree D_c contains all odd numbers such that the length of Collatz computation for n is c. It suffices to show that the union of the trees D_c is a tree.

Lemma 3.1. For every element n of the graph $\mathcal{G}$ such that $n \mod 3 \neq 0$ if there exist three natural numbers such that the following equality $n \cdot 3^x + y = 2^z$ holds, then for every positive, natural number $i > 0$ there exist three numbers x_i, y_i, z_i such that the equality $S_i(n) \cdot 3^{x_i} + y_i = 2^{z_i}$ holds.

Example 3.1. A few subtrees contained in the graph $\mathcal{G}$

c	$D_c = \langle V_c, E_c \rangle$	numbers on diagonal c
0	$\langle\{1\}, \emptyset\rangle$	1
5	$\langle\{1,5\}, \{(1,5)\}\rangle$	5
7	$\langle\{1,5,21,3\}, \{(1,5),(1,21),(5,3)\}\rangle$	$21, 3$
9	$\langle\{1,5,21,,85,113\}, \{(1,5),(1,21),(5,3),(1,85),(5,13)\}\rangle$	$85, 13$
11	$\langle\{1,5,21,3,85,13,341,53\}, \{(1,5),(1,21),(5,3),(1,85),(5,13),(1,341),(5,53)\}\rangle$	$341, 53$
12	$\langle\{1,5,\ldots,17,113\}, \{(1,5),\ldots,(85,113),(13,17)\}\rangle$	$13, 17$

The trees form the increasing sequence $D_0 \subset D_5 \subset D_9 \subset D_{11} \subset D_{12} \subset D_{13} \subset D_{14}$.
Every tree is bounded by a diagonal $c = x + z$.
The figure 4 illustraes the trees $D_0 - D_{14}$.

Notice that each tree D_c, is the lower-left part of the graph G. For every natural number c the tree D_c contains all odd

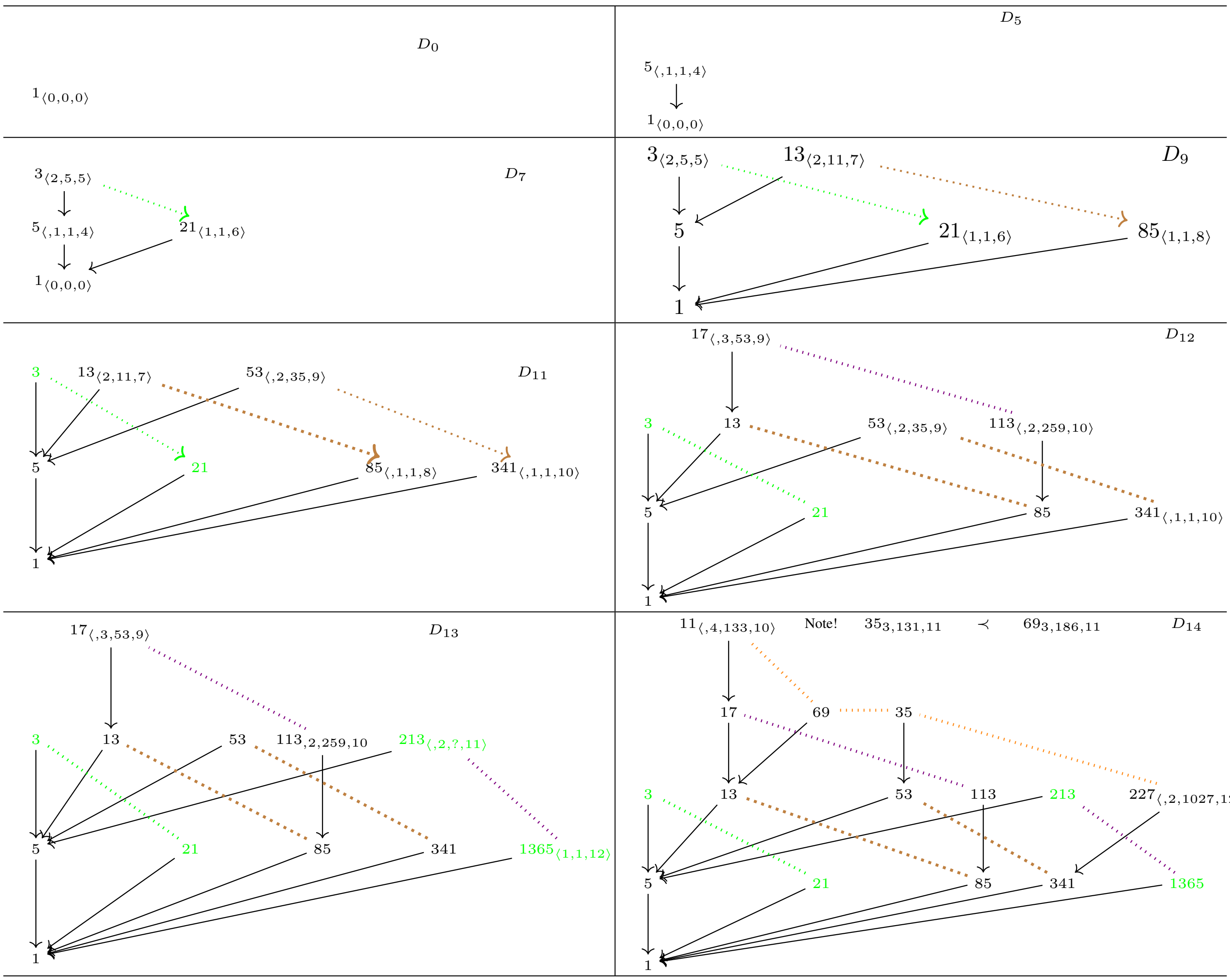

Figure 4. Couple of trees D_c

numbers that are not farther than c steps from 1. It means that for every node $o \in V_c$ the computation has no more than $c = x + z$ steps and the equation $n \cdot 3^x + y = 2^z$ holds.

End of Example 3.1

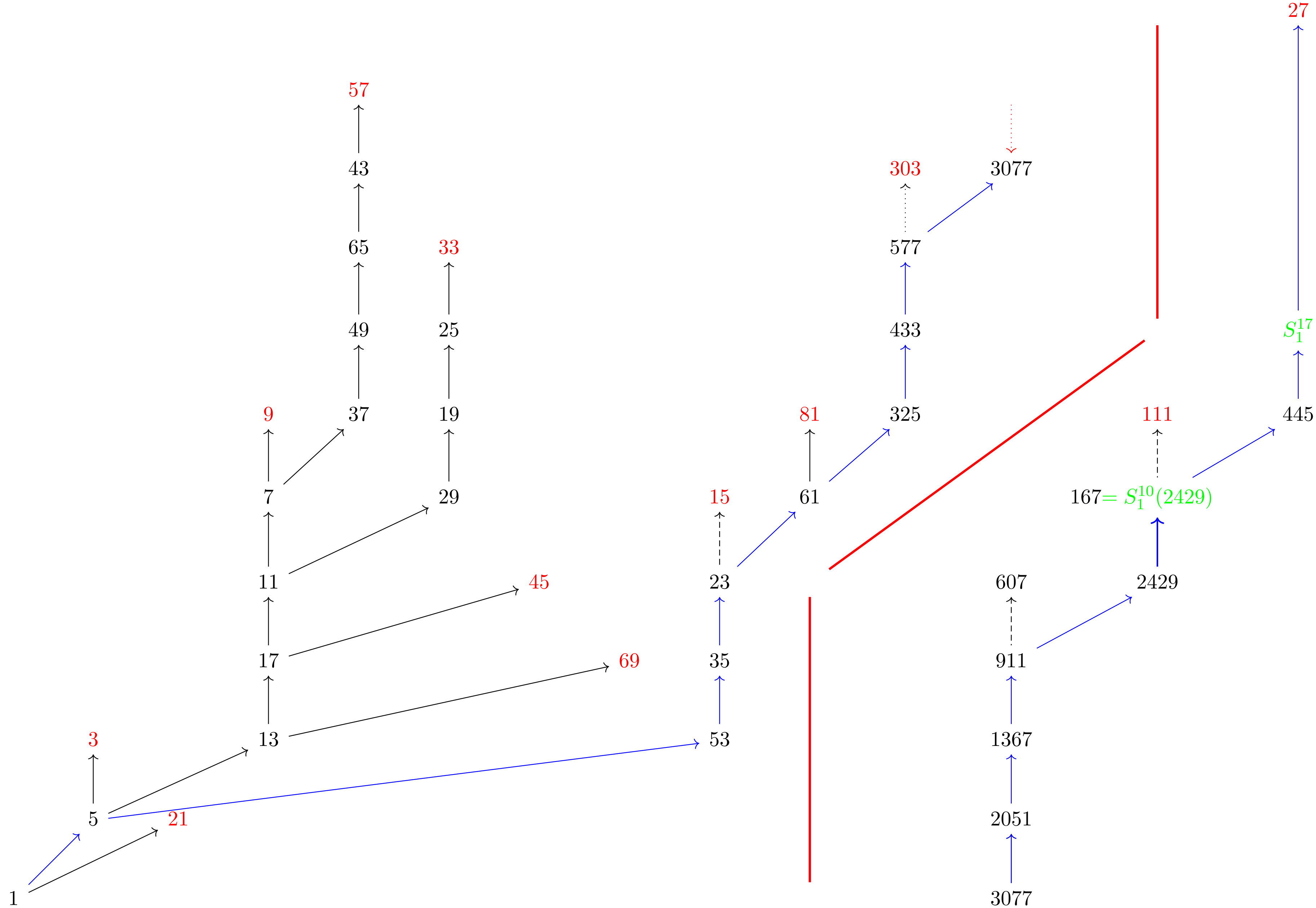

Figure 5. A subgraph of the graph $\mathcal{G}$ from another perspective $27 = S_1^{17}(S_2(S_1^{10}(S_2(S_1^{3}(3077)))))$ and $3077 = S_2(S_1^{2}(S_2^{2}(S_1^{2}(S_3(2(1))))))$

Lemma 3.2. The trees D_c have the following properties

(*i*) The trees $D_0, D_5, D_7, D_9, D_{11}$ are defined as shown in the example 3.1.

(*ii*) Trees $D_1, D_2, D_3, D_4, D_6, D_8, D_{10}$ are not defined.

(*iii*) For every $c \geq 11$ the following inclusion holds $D_c \subsetneq D_{c+1}$ For $c+1 = 5, 7, 9, 11$ trees D_{c+1} are constructed on the base of trees $D_1, D_5, , D_7, D_9$ respectively.

(*iv*) For every tree D_c and for every number $n \in D_c$ if a successor $S_i(n)$ has a path of length $c+1$ then the number n is in the tree D_c, i.e. $n \in D_c$

We can use these observations to

- define an ordering $\prec$ of odd natural numbers. We shall order odd, natural numbers in a way similar to the rdering pairs of natural numbers defined by G. Cantor.

  Definition 3.1. Let a and b be two odd, natural numbers. We assume that the number c is the least natural number that a is in the tree D_c. Similarly, we assume that $b \in D_d$ and for every number $d' < d$ the relation $b \notin D_{d'}$ holds. By the lemma 3.1 we know that exists two triples $\langle x, y, z\rangle$ and $\langle u, v, t\rangle$ sch that two equations hold $a \cdot 3^x + y = 2^z$ and $b \cdot 3^u + v = 3^t$.
  We put
  $$a \prec b \overset{df}{\equiv} x + z < u + t \vee x + z = u + t \wedge x < u \vee x + z = u + t \wedge x = u \wedge y < v \tag{12}$$

  Example 3.2. An initial segment of the sequence of natural numbers as ordered by the $\prec$ relation
  $1 \prec 5 \prec 21 \prec 3 \prec 85 \prec 13 \prec 341 \prec 53 \prec 113 \prec 17 \prec 1365 \prec 213 \prec 227 \prec 35 \prec 69 \prec 11 \prec \ldots$

- to prove that if one draws the hotel Collatz putting its columns in accordance withe relation $\prec$ then all edges goto the left, hence the graph $\mathcal{HC}$, Fig. 7, is **tree!**

4. T3 – the structure of the proof

Our proof is conducted in a formalized algorithmic theory $\mathcal{ATN}$ of natural numbers. Algorithmic theories *differ* from elementary theories. A system $\langle \mathcal{L}, \mathcal{C}, \mathcal{A}x\rangle$ is xalled algorithmic theory if the language $\mathcal{L}$ contains three subsets of the set of well formed expressions: *terms, formulas, and programs aka algorithms*. The logical consequence operator gets three infinitary inference rules, Three non-logical axioms of the theory are:

$$\begin{array}{ll} Ax_1) & x + 1 \neq 0 \\ Ax_2) & x \neq y \implies x + 1 \neq y + 1 \\ Ax_3) & \{q := 0;\ \mathbf{while}\ x \neq q\ \mathbf{do}\ q := q + 1\ \mathbf{od}\}(x = q) \end{array}$$

The following diagram shows the structure of the proof of main theorem.

$$St_0 \stackrel{df}{=} \left\{ \boxed{ n = \underline{r} \implies \left(\overbrace{\begin{array}{l} \left(r \cdot 3^{i_r} + y(i_r) = 2^{z_{i_r}}\right) \wedge \\ \left(y(i_r) = \sum_{j=0}^{i_r - 1} 3^{i_r-1-j} \cdot 2^{\sum_{p=0}^{j} k_p(r)}\right) \wedge \\ \left(z_{i_r} = \sum_{j=0}^{i_r} k_j(r)\right) \end{array}}^{\Psi(r,i_r)} \right) } \right\}_{r=1}^{\infty}$$

$$\Big\downarrow \text{Lemma 5.1}$$

$$St_1 \stackrel{df}{=} \left\{ \boxed{ n = \underline{r} \implies \{\Gamma\}\,\{\mathbf{if}\ m \neq 1\ \mathbf{then}\ \Delta\ \mathbf{fi}\}^{i(\underline{r})}\,(m = 1) } \right\}_{r=1}^{\infty}$$

$$\Big\downarrow$$

$$St_2 \stackrel{df}{=} \left\{ \boxed{ n = \underline{r} \implies \{\Gamma;\ \mathbf{while}\ m \neq 1\ \mathbf{do}\ \Delta\ \mathbf{od}\}\,(m = 1) } \right\}_{r=1}^{\infty}$$

$$\Big\downarrow\ r{=}0{+}\underbrace{1+1+\cdots+1}_{r\ \times}$$

$$St_3 \stackrel{df}{=} \left\{ \boxed{ \left\{ \begin{array}{l} q := 0; \\ \left\{\ q := q + 1\ \right\}^{\underline{r}} \end{array} \right\} (n = q) \implies \left\{ \begin{array}{l} \Gamma; \\ \mathbf{while}\ m \neq 1 \\ \mathbf{do}\ \Delta\ \mathbf{od} \end{array} \right\} (m = 1) } \right\}_{r=1}^{\infty}$$

$$\Big\downarrow \text{applying the rule R3}$$

$$\forall_{n \neq 0} \underbrace{\boxed{ \left\{ \begin{array}{l} q := 1; \\ \mathbf{while}\ n \neq q\ \mathbf{do} \\ \quad q := q + 1 \\ \mathbf{od} \end{array} \right\} (n = q) }}_{\mathbb{FORALL}\ n,\ \mathbb{IF}\ n\ is\ a\ natural\ number} \implies \underbrace{\boxed{ \left\{ \begin{array}{l} m := n \div 2^{\kappa(n)}; \\ \mathbf{while}\ m \neq 1\ \mathbf{do} \\ \quad m := 3 \cdot m + 1; \\ \quad m := m \div 2^{\kappa(m)} \\ \mathbf{od} \end{array} \right\} (m = 1) }}_{\mathbb{THEN}\ the\ computation\ for\ n\ is\ finite\ \mathbb{FI}}$$

$$\Big\downarrow$$

$$\mathcal{ATN} \vdash \forall_{n \neq 0} \boxed{ \left\{ \begin{array}{l} m := n \div 2^{\kappa(n)}; \\ \mathbf{while}\ m \neq 1\ \mathbf{do} \\ \quad m := 3 \cdot m + 1; \\ \quad m := m \div 2^{\kappa(m)} \\ \mathbf{od} \end{array} \right\} (m = 1) }$$

Figure 6. The structure of the proof of Main theorem

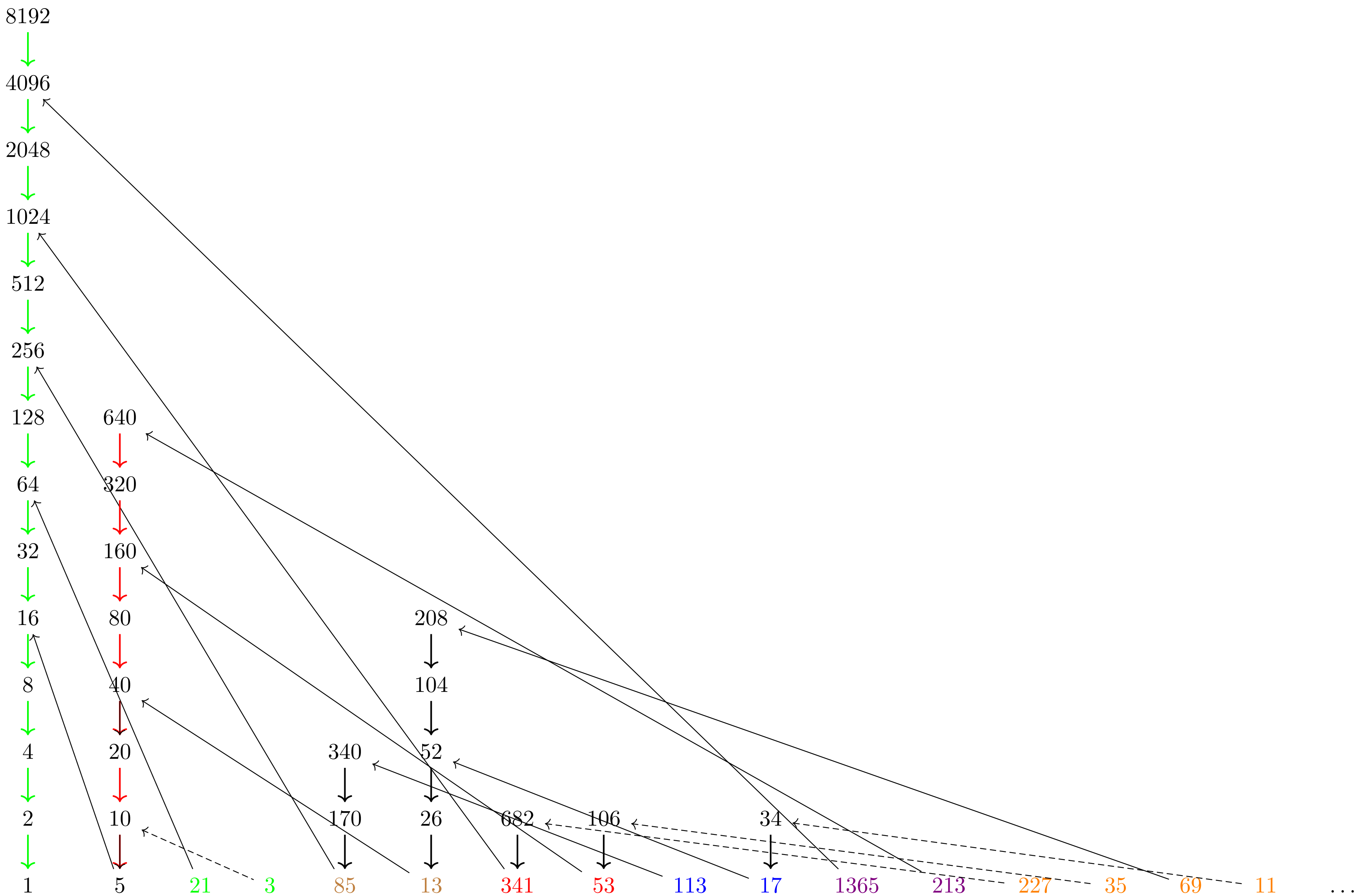

Figure 7. Hotel Collatz with columns ordered by the relation $\prec$. Note, all diagonal arrows point to the left.